Title	Local Torelli Theorem for some non-singular weighted complete intersections
Author(s)	Usui, Sampei
Citation	Proceedings of the International Symposium on Algebraic Geometry : Kyoto, 1977. p.723-p.734
Issue Date	1978
oaire:version	VoR
URL	https://hdl.handle.net/11094/73374
rights	
Note	

Intl. Symp. on Algebraic Geometry
Kyoto, 1977, pp. 723–734

Local Torelli theorem for some non-singular weighted complete intersections

By Sampei Usui

(Received June 3, 1976)

Introduction

For a compact Kähler manifold X, P. A. Griffiths constructed and studied a "period map" from the parameter space of the deformations of X to the "Griffiths domain" which parametrizes the Hodge structures on the underlying differentiable manifold of X ([2]). As one of these studies, he formulated the local Torelli problem in the form (P_1) in § 1, which asks whether a period map separates infinitely near points (cf. [2] and also [7]). After examining this problem in some examples, Griffiths presented in his article [3] a problem: "(7.1). PROBLEM*. Find methods to treat the local Torelli theorem. In particular, decide whether it is true or false for simply-connected canonical surfaces (i.e. surfaces with ample canonical bundle)."

The purpose of the present paper is to give a proof of the local Torelli theorem for some non-singular "weighted complete intersections" (see Theorem 2.1 below).

In the author's previous paper [7], he gave a proof of the local Torelli theorem for non-singular complete intersections with ample canonical line bundle. On the other hand, Sigefimi Mori introduced and studied a family of varieties so called "weighted complete intersections" ([5]). Weighted complete intersections have several properties similar to those of complete intersections (cf. [5] and also § 1), so that our method in [7] is naturally applicable for some weighted complete intersections. However, we need some assumptions (the assumptions i) and ii) in Theorem 2.1) in order to let our method work well.

§ 3 includes the discussions on these assumptions and some examples for which our method breaks down.

The author wishes to thank Professor S. Mori for the valuable suggestions.

§ 1. Preliminaries

a. Local Torelli porblem. In this subsection, we recall the formulation of the local Torelli problem (for the background of this problem see [2] and also [7]).

Let X be a compact Kähler manifold. The local Torelli problem is the question whether the period map of Griffiths separates infinitely near points at the point, in the parameter space of the Kuranishi family of the deformations of X, corresponding to X.

The local Torelli problem in r-th cohomology is formulated as follows.

(P_1) *Let* $H^1(X, T_X)\otimes H^{r-i+1,i-1}(X)\to H^{r-i,i}(X)$ *be the pairing induced from* $T_X\otimes\Omega_X^{r-i+1}\to\Omega_X^{r-i}$ $(1\leq i\leq r)$. *Is* $\tau\in H^1(X, T_X)$ *zero if* $\tau\cdot\alpha_i$ *is zero in* $H^{r-i,i}(X)$ *for every* $\alpha_i\in H^{r-i+1,i-1}(X)$ $(1\leq i\leq r)$?

b. Weighted complete intersections. S. Mori generalized the notion of complete intersection and studied so called "weighted complete intersections" ([5]).

The following definitions and results are found in [5].

Definition 1.1. Let $n, e_0, \cdots, e_n$ be positive integers. We set

$$m=\text{l.c.m.}\{e_0, \cdots, e_n\},$$

and

$$r(e)=r(e_0, \cdots, e_n)=\min_{p:\ \text{prime}} \#\{i\,|\,0\leq i\leq n, p\nmid e_i\}.$$

Let K be a field.

$\boldsymbol{Q}(e_0, \cdots, e_n)$, or simply $\boldsymbol{Q}(e)$ is the scheme $\operatorname{Proj}(K[X_0, \cdots, X_n])$, where the gradation of $K[X_0, \cdots, X_n]$ is defined as follows;

$$\deg X_i=e_i\ (0\leq i\leq n),\quad \text{and}\quad \deg a=0\ (a\in K).$$

For an integer a, $\mathcal{O}_{Q(e)}(a)$, or simply $\mathcal{O}_Q(a)$ is the coherent $\mathcal{O}_{Q(e)}$-module corresponding to the homogeneous $K[X_0, \cdots, X_n]$-module $K[X_0, \cdots, X_n](a)$.

For a positive integer h, S_h is the closed subset of $\boldsymbol{Q}(e)$, defined by the ideal generated by $\{X_i\,|\,h\nmid e_i\}$.

It is easy to see that the singular locus of $\boldsymbol{Q}(e)$ is $\bigcup_{1<h} S_h$.

Definition 1.2. With the notations in Definition 1.1, $\boldsymbol{P}(e_0, \cdots, e_n)$, or simply $\boldsymbol{P}(e)$ is the open subscheme $\boldsymbol{Q}(e)-\bigcup_{1<h} S_h$ of $\boldsymbol{Q}(e)$. We call the scheme $\boldsymbol{P}(e)$ a weak projective space of size $(e_0, \cdots, e_n)$, or simply of size (e). We define $\mathcal{O}_P(a)=\mathcal{O}_Q(a)\,|\,\boldsymbol{P}(e)$ for every integer a.

Definition 1.3. With the notations in Definitions 1.1 and 1.2, let $c, a_1, \cdots, a_c$ be positive integers. We consider $\operatorname{Proj}(K[X_0, \cdots, X_n]/(F_1, \cdots, F_c))$ for homogeneous elements $F_1, \cdots, F_c$ in the graded ring $K[X_0, \cdots, X_n]$ with $\deg F_j=a_j$ $(1\leq j\leq c)$, satisfying the following conditions.

i) $F_1, \cdots, F_c$ form a regular sequence of $K[X_0, \cdots, X_n]$.

ii) $V_+(F_1, \cdots, F_c)\cap\bigcup_{1<h} S_h=\varnothing$.

An algebraic K-scheme X is called a weighted complete intersection in $\boldsymbol{P}(e)$ of type $(a_1, \cdots, a_c)$, if X is isomorphic to such a K-scheme

$$\mathrm{Proj}\,(K[X_0, \cdots, X_n]/(F_1, \cdots, F_c)).$$

In this case, for an arbitrary integer a, we denote by $\mathcal{O}_X(a)$ the $\mathcal{O}_X$-module induced from $\mathcal{O}_P(a)$.

Results. (1) *We see that* $\mathrm{codim}_{Q(e)} \bigcup_{1<h} S_h = r(e)$, *and the maximal dimension of complete subschemes of* $\boldsymbol{P}(e)$ *is given by* $r(e)-1$.

(2) $\mathcal{O}_P(1)$ *is an invertible sheaf on* $\boldsymbol{P}(e)$, *and we have a natural isomorphism* $\mathcal{O}_P(1)^{\otimes a} \xrightarrow{\sim} \mathcal{O}_P(a)$ *for every integer* a.

(3) *For a weighted complete intersection* X, $\mathcal{O}_X(1)$ *is an ample invertible sheaf. Moreover, if* $\dim X \geq 3$, *then we have* $\mathrm{Pic}\, X \approx \boldsymbol{Z}$ *and* $\mathcal{O}_X(1)$ *generates* $\mathrm{Pic}\, X$.

(4) *Let* $X = \mathrm{Proj}\,(K[X_0, \cdots, X_n]/(F_1, \cdots, F_c))$ *be a weighted complete intersection of dimension* ≥ 1 *in* $\boldsymbol{P}(e)$ *of type* $(a_1, \cdots, a_c)$. *Then we have*

$$(K[X_0, \cdots, X_n]/(F_1, \cdots, F_c))_a \xrightarrow{\sim} H^0(X, \mathcal{O}_X(a)) \qquad (a \in \boldsymbol{Z}),$$
$$H^j(X, \mathcal{O}_X(a)) = 0 \qquad (a, j \in \boldsymbol{Z}, 0<j<n-c=\dim X),$$

and

$$\omega_X = \mathcal{O}_X\left(\sum_{j=1}^{c} a_j - \sum_{i=0}^{n} e_i\right)$$

where ω_X *denotes the dualizing sheaf of* X.

(5) *We have the following exact sequence.*

$$0 \longrightarrow \mathcal{O}_P \longrightarrow \oplus_{i=0}^{n} \mathcal{O}_P(e_i) \longrightarrow T_P \longrightarrow 0.$$

Restricting this sequence to a weighted complete intersection X *in* $\boldsymbol{P}(e)$, *we have*

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \oplus_{i=0}^{n} \mathcal{O}_X(e_i) \longrightarrow T_P | X \longrightarrow 0.$$

(6) *If* X *is a non-singular weighted complete intersection in* $\boldsymbol{P}(e)$ *of type* $(a_1, \cdots, a_c)$, *then the normal bundle* $N_{X/P}$ *of* X *in* $\boldsymbol{P}(e)$ *is isomorphic to* $\oplus_{j=1}^{c} \mathcal{O}_X(a_j)$, *and we have the following exact sequence.*

$$0 \longrightarrow T_X \longrightarrow T_P | X \longrightarrow N_{X/P} \longrightarrow 0.$$

§ 2. Local Torelli theorem

In this section, we shall apply the method used in the previous paper [7] to non-singular weighted complete intersections.

We use the same notations as in § 1. We take $K=\boldsymbol{C}$ the field of complex numbers.

The following theorem is our main result.

Theorem 2.1. *Let* $X = \mathrm{Proj}\,(\boldsymbol{C}[X_0, \cdots, X_n]/(F_1, \cdots, F_c))$ *be a non-singular weighted complete intersection in* $\boldsymbol{P}(e_0, \cdots, e_n)$ *of type* $(a_1, \cdots, a_c)$ *of dimension*

$d \geq 2$. Assume that $a_j > 1$ ($1 \leq j \leq c$), and that the canonical line bundle is ample. Assume, furthermore,

i) $e_0 = e_1 = 1$;

ii) *for some integer b with $2 \leq b \leq n$, $F_1, \cdots, F_c, X_0, X_1, X_b$ form a regular sequence of $C[X_0, \cdots, X_n]$.*

Then the local Torelli theorem in the d-th cohomology holds for X.

Before proving the above theorem, we first reformulate the local Torelli problem for such a non-singular weighted complete intersection X as in Theorem 2.1.

In addition to the notations in § 1, we use the following notations; $R = C[X_0, \cdots, X_n]$, I is the homogeneous ideal of R generated by $\{F_1, \cdots, F_c\}$, and k is the integer with $\mathcal{O}_X(k) = K_X$ (cf. (4) in § 1).

From the exact sequence (5) in § 1, we have the exact sequence of cohomology groups:

$$\oplus_{i=0}^n H^1(X, \mathcal{O}_X(e_i)) \longrightarrow H^1(X, T_P|X) \longrightarrow H^2(X, \mathcal{O}_X) \longrightarrow \oplus_{i=0}^n H^2(X, \mathcal{O}_X(e_i)).$$

By (4) in § 1, we see that $\oplus_{i=0}^n H^1(X, \mathcal{O}_X(e_i)) = 0$, and that $H^2(X, \mathcal{O}_X) = 0$ in case $d > 2$ and $H^2(X, \mathcal{O}_X) \to \oplus_{i=0}^n H^2(X, \mathcal{O}_X(e_i))$ is injective in case $d = 2$. Indeed, since $k > 0$, the last statement follows from the fact that the dual of this morphism $\oplus_{i=0}^n H^0(X, \mathcal{O}_X(k - e_i)) \to H^0(X, \mathcal{O}_X(k))$ is surjective (cf. (4) in § 1). Hence we have $H^1(X, T_P|X) = 0$. Therefore, from the exact sequences (5) and (6) in § 1, we have the following diagram.

$$\begin{array}{ccccccc}
\oplus_{i=0}^n H^0(X, \mathcal{O}_X(e_i)) & & & & & & \\
\downarrow & & & & & & \\
H^0(X, T_P|X) & \longrightarrow & \oplus_{j=1}^c H^0(X, \mathcal{O}_X(a_j)) & \longrightarrow & H^1(X, T_X) & \longrightarrow & 0, \\
\downarrow & & & & & & \\
0 & & & & & &
\end{array}$$

where the row and the column are exact. Tensoring K_X to the exact sequences (5) and (6) in § 1, we obtain a similar diagram.

$$\begin{array}{ccccc}
\oplus_{i=0}^n H^0(X, \mathcal{O}_X(e_i + k)) & & & & \\
\downarrow & & & & \\
H^0(X, T_P(k)|X) & \longrightarrow & \oplus_{j=1}^c H^0(X, \mathcal{O}_X(a + k)) & \longrightarrow & H^1(X, T_X(k)), \\
\downarrow & & & & \\
0 & & & &
\end{array}$$

where the row and the column are exact. Note that the composite homomorphism $\oplus_{i=0}^n H^0(X, \mathcal{O}_X(e_i + \ell)) \to H^0(X, T_P(\ell)|X) \to \oplus_{j=1}^c H^0(X, \mathcal{O}_X(a_j + \ell))$ ($\ell = 0, k$) is given by the Jacobian matrix

$$\begin{bmatrix} \partial_0 F_1 \cdots \partial_n F_1 \\ \cdots\cdots\cdots \\ \partial_0 F_c \cdots \partial_n F_c \end{bmatrix},$$

where $\partial_i F_j$ denotes $\partial F_j / \partial X_i$ $(1 \leq j \leq c, 0 \leq i \leq n)$. Therefore, in order to prove (P_1) for $r=d$, it is enough to prove the following (P_2).

(P_2) *Let Φ_j be homogeneous elements in R of degree a_j $(1 \leq j \leq c)$. Suppose that, for each homogeneous element G in R of degree k, there exist homogeneous elements $A_0, \cdots, A_n$ in R satisfying*

$$G \begin{bmatrix} \Phi_1 \\ \vdots \\ \Phi_c \end{bmatrix} \equiv \begin{bmatrix} \partial_0 F_1 \cdots \partial_n F_1 \\ \cdots\cdots\cdots \\ \partial_0 F_c \cdots \partial_n F_c \end{bmatrix} \begin{bmatrix} A_0 \\ \vdots \\ A_n \end{bmatrix} \quad \bmod I.$$

Then there exist homogeneous elements $B_0, \cdots, B_n$ in R satisfying

$$\begin{bmatrix} \Phi_1 \\ \vdots \\ \Phi_c \end{bmatrix} \equiv \begin{bmatrix} \partial_0 F_1 \cdots \partial_n F_1 \\ \cdots\cdots\cdots \\ \partial_0 F_c \cdots \partial_n F_c \end{bmatrix} \begin{bmatrix} B_0 \\ \vdots \\ B_n \end{bmatrix} \quad \bmod I.$$

In the statement (P_2) above, we take G to be a monomial only in X_0, X_1. Then, by splitting G into a product of X_ν $(\nu=0, 1)$ and another monomial and by multiplying the latter to Φ_j's, we modify (P_2) a little stronger assertion as follows.

(P_3) *Let Φ_j be homogeneous elements in R of degree $\leq a_j+k-1$ $(1 \leq j \leq c)$. Suppose that, for each ν $(\nu=0, 1)$, there exist homogeneous elements $A_{\nu 0}, \cdots, A_{\nu n}$ in R with $\deg A_{\nu i} \leq k+e_i$ $(0 \leq i \leq n)$ satisfying*

$$(*_\nu) \qquad X_\nu \begin{bmatrix} \Phi_1 \\ \vdots \\ \Phi_c \end{bmatrix} \equiv \begin{bmatrix} \partial_0 F_1 \cdots \partial_n F_1 \\ \cdots\cdots\cdots \\ \partial_0 F_c \cdots \partial_n F_c \end{bmatrix} \begin{bmatrix} A_{\nu 0} \\ \vdots \\ A_{\nu n} \end{bmatrix} \quad \bmod I.$$

Then there exist homogeneous elements $B_0, \cdots, B_n$ in R satisfying

$$\begin{bmatrix} \Phi_1 \\ \vdots \\ \Phi_c \end{bmatrix} \equiv \begin{bmatrix} \partial_0 F_1 \cdots \partial_n F_1 \\ \cdots\cdots\cdots \\ \partial_0 F_c \cdots \partial_n F_c \end{bmatrix} \begin{bmatrix} B_0 \\ \vdots \\ B_n \end{bmatrix} \quad \bmod I.$$

It is easy to see that, by the inductive use of (P_3), we obtain (P_2).

Next we prove some lemmas which play an essential roll in the proof of Theorem 2.1.

Lemma 2.2. *With the notations in Theorem* 2.1, *we have*

$$H^0(X, \Omega_X^{d-1}(\ell))=0 \qquad (\ell \leq 1).$$

Proof. We prove this in two steps.

Step 1. $H^0(X, \Omega_P^{d-1}(\ell)\,|\,X) \xrightarrow{\sim} H^0(X, \Omega_X^{d-1}(\ell))$ $(\ell \leq 1)$.

Indeed, we have the following exact sequence dual to the exact sequence (6) in § 1.

$$(1) \qquad 0 \longrightarrow \check{N} \longrightarrow \Omega_P^1\,|\,X \longrightarrow \Omega_X^1 \longrightarrow 0,$$

where $\check{N}$ is the conormal bundle of X in $\boldsymbol{P}(e)$. By (6) in § 1, we have $\check{N} = \oplus_{j=1}^c \mathcal{O}_X(-a_j)$. The exact sequence (1) induces a filtration of $\Omega_P^{d-1}\,|\,X$. Tensoring $\mathcal{O}_X(\ell)$, we have the filtration

$$\Omega_P^{d-1}(\ell)\,|\,X = E_0 \supset E_1 \supset \cdots \supset E_{d-1} \supset E_d = 0$$

of $\Omega_P^{d-1}(\ell)\,|\,X$, with the successive quotients

$$G_i = E_i / E_{i+1} \approx \left(\bigwedge^i \check{N}\right) \otimes \Omega_X^{d-1-i}(\ell).$$

Hence we have the following short exact sequences.

$$(2) \qquad 0 \longrightarrow E_{i+1} \longrightarrow E_i \longrightarrow G_i \longrightarrow 0 \qquad (0 \leq i \leq d-1).$$

Since $a_j > 1$ $(1 \leq j \leq c)$,

$$\left(\bigwedge^i \check{N}\right) \otimes \mathcal{O}_X(\ell) \qquad (i > 0,\ \ell \leq 1)$$

are direct sums of negative line bundles. Hence, by the Nakano vanishing theorem, we have

$$H^j(X, G_i) = 0 \qquad (i > 0,\ j = 0, 1).$$

Therefore, observing the cohomology sequences of the exact sequences (2), we obtain the desired results.

Step 2. $H^0(X, \Omega_P^{d-1}(\ell)\,|\,X) = 0$ $(\ell \leq 1)$.

We have the following exact sequence dual to the exact sequence (5) in § 1.

$$(3) \qquad 0 \longrightarrow \Omega_P^1\,|\,X \longrightarrow \oplus_{i=0}^n \mathcal{O}_X(-e_i) \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

In case $d=2$, tensoring $\mathcal{O}_X(\ell)$ to (3), we have

$$(4) \qquad 0 \longrightarrow \Omega_P^1(\ell)\,|\,X \longrightarrow \oplus_{i=0}^n \mathcal{O}_X(\ell - e_i) \longrightarrow \mathcal{O}_X(\ell) \longrightarrow 0.$$

For $\ell \leq 0$, we see that $H^0(X, \mathcal{O}_X(\ell - e_i)) = 0$. Hence $H^0(X, \Omega_P^1(\ell)|X) = 0$. For $\ell = 1$, we see that

$$H^0(X, \mathcal{O}_X(1-e_i)) \cong \begin{cases} R_0 & \text{if } e_i = 1, \\ 0 & \text{if } e_i > 1, \end{cases}$$

and that

$$H^0(X, \mathcal{O}_X(1)) \cong R_1.$$

Hence we have

$$\begin{array}{ccc} \oplus_{i=1}^n H^0(X, \mathcal{O}_X(1-e_i)) & \longrightarrow & H^0(X, \mathcal{O}_X(1)) \\ \wr\uparrow & & \wr\uparrow \\ R_0 \oplus \cdots \oplus R_0 & \xrightarrow{\sim} & R_1. \end{array}$$

Therefore, by the exact sequence (4), we obtain the desired results in this case. In case $d \geq 3$, observe the following exact sequence induced from (3).

$$0 \longrightarrow \Omega_P^{d-1}(\ell)|X \longrightarrow \left(\bigwedge^{d-1} (\oplus_{i=0}^n \mathcal{O}_X(-e_i)) \right) \otimes \mathcal{O}_X(\ell).$$

Since

$$H^0\left(X, \left(\bigwedge^{d-1} (\oplus_{i=0}^n \mathcal{O}_X(-e_i)) \right) \otimes \mathcal{O}_X(\ell) \right) = 0 \qquad (\ell \leq 1),$$

we obtain the desired results. Q.E.D.

By virtue of Lemma 2.2, we obtain the following key lemma.

Lemma 2.3. *Let A_i be homogeneous elements in R with* $\deg A_i \leq k + e_i + 1$ $(0 \leq i \leq n)$. *Assume that*

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \equiv \begin{bmatrix} \partial_0 F_1 & \cdots & \partial_n F_1 \\ & \cdots\cdots & \\ \partial_0 F_c & \cdots & \partial_n F_c \end{bmatrix} \begin{bmatrix} A_0 \\ \vdots \\ A_n \end{bmatrix} \qquad \bmod I.$$

Then there exists a homogeneous element C in R, which is independent of the suffix i, such that

$$A_i \equiv e_i X_i C \quad \bmod I \qquad (0 \leq i \leq n).$$

Proof. From the exact sequences (5) and (6) in § 1, we have the following diagram:

$$
\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \downarrow & & & \\
 & & & H^0(X, \mathcal{O}_X(\ell')) & & & \\
 & & & \downarrow & & & \\
 & & & \bigoplus_{i=0}^{n} H^0(X, \mathcal{O}_X(e_i+\ell')) & & & \\
 & & & \downarrow{\scriptstyle\alpha} & & & \\
0 \longrightarrow & H^0(X, T_X(\ell')) & \longrightarrow & H^0(X, T_P(\ell')\,|\,X) & \xrightarrow{\beta} & H^0(X, N_{X/P}(\ell')). & \\
 & & & \downarrow & & & \\
 & & & 0 & & &
\end{array}
$$

Since the composite homomorphism $\beta\cdot\alpha$ is given by the Jacobian matrix

$$\begin{bmatrix} \partial_0 F_1 \cdots \partial_n F_1 \\ \cdots\cdots\cdots \\ \partial_0 F_c \cdots \partial_n F_c \end{bmatrix},$$

the statement of Lemma 2.3 is equivalent to

$$(5) \qquad H^0(X, T_X(\ell'))=0 \qquad (\ell' \leq k+1).$$

Moreover, since $T_X \otimes K_X \xrightarrow{\sim} \Omega_X^{q-1}$, that is, $T_X(k)=\Omega_X^{q-1}$, (5) is equivalent to Lemma 2.2. Q.E.D.

Finally we prove Theorem 2.1.

Proof. It is enough to prove the assertion (P_3). Let $X_0(*_1)$ (resp. $X_1(*_0)$) be the equation obtained by multiplying X_0 (resp. X_1) to the equation $(*_1)$ (resp. $(*_0)$) in (P_3). By subtracing the equation $X_1(*_0)$ from the equation $X_0(*_1)$, we have that

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \equiv \begin{bmatrix} \partial_0 F_1 \cdots \partial_n F_1 \\ \cdots\cdots\cdots \\ \partial_0 F_c \cdots \partial_n F_c \end{bmatrix} \begin{bmatrix} X_0 A_{10} - X_1 A_{00} \\ \vdots \\ X_0 A_{1n} - X_1 A_{0n} \end{bmatrix} \qquad \operatorname{mod} I.$$

Applying Lemma 2.3 for $A_i = X_0 A_{1i} - X_1 A_{0i}$, we obtain C in R such that

$$(6) \qquad X_0 A_{1i} - X_1 A_{0i} \equiv e_i X_i C \quad \operatorname{mod} I \qquad (0 \leq i \leq n).$$

We now use the assumption ii) repeatedly in order to cancel X_b, X_1, X_0. Observing the above formula (6) for $i=b$ modulo $(F_1, \cdots, F_c, X_0, X_1)$, we can cancel X_b and we have

$$C \equiv X_0 D + X_1 E \qquad \operatorname{mod} I$$

for some D and E in R. Substituting this formula into (6), we have

$$X_0(A_{1i}-e_iX_iD)-X_1(A_{0i}+e_iX_iE)\equiv 0 \qquad \mathrm{mod}\, I.$$

Observing this formula modulo $(F_1, \cdots, F_c, X_0)$, we can cancel X_1 and we obtain

$$(7) \qquad A_{0i}+e_iX_iE\equiv X_0B_i \qquad \mathrm{mod}\, I$$

for some B_i in R $(0\leq i\leq n)$. Substituting this formula (7) into $(*_0)$ in (P_3), we have

$$X_0\begin{bmatrix}\Phi_1\\ \vdots\\ \Phi_c\end{bmatrix}\equiv X_0\begin{bmatrix}\partial_0F_1\cdots\partial_nF_1\\ \cdots\cdots\\ \partial_0F_c\cdots\partial_nF_c\end{bmatrix}\begin{bmatrix}B_0\\ \vdots\\ B_n\end{bmatrix}-E\begin{bmatrix}\partial_0F_1\cdots\partial_nF_1\\ \cdots\cdots\\ \partial_0F_c\cdots\partial_nF_c\end{bmatrix}\begin{bmatrix}e_0X_0\\ \vdots\\ e_nX_n\end{bmatrix} \qquad \mathrm{mod}\, I.$$

The last term in this equation vanishes by the Euler's formula. Observing this equation modulo $(F_1, \cdots, F_c)$, we can cancel X_0 and we obtain the desired conclusion. Q.E.D.

§ 3. On the assumptions i) and ii) in Theorem 2.1

In this section we discuss the additional assumptions i) and ii) in Theorem 2.1. We use the same notations as in § 1.

In the following cases, the assumptions i) and ii) in Theorem 2.1 are fulfiled.

Proposition 3.1. *Let* $X=\mathrm{Proj}\,(K[X_0, \cdots, X_n]/(F_1, \cdots, F_c))$ *be a weighted complete intersection of dimension*≥ 2 *in* $\boldsymbol{P}(e)$. *If one of the following conditions is satisfied, then the assumption* ii) *in Theorem* 2.1 *is fulfiled.*

i) $e_0=e_1=1$, *and that* $\dim X$ *is the maximal dimension of the complete subschemes of* $\boldsymbol{P}(e)$.

ii) $e_0=e_1=\cdots=e_{n-1}=1$. *In this case, we may have to change* $X_0, \cdots, X_{n-1}$ *by a linear transformation if necessary.*

Proof. *Case i).* Let S_h be a maximal dimensional component of the singular locus $\bigcup_{1<h} S_h$ of $\boldsymbol{Q}(e)$. We see that $S_h=V_+(\{X_i \mid h\nmid e_i\})$ and that $\#\{X_i \mid h\nmid e_i\}=r(e)$ by definition. We have $\dim X=r(e)-1$ by virtue of (1) in § 1. Hence we have that

$$\#(\{F_1, \cdots, F_c\} \coprod \{X_i \mid h\nmid e_i\})=n+1.$$

On the other hand, by the definition of weighted complete intersection, we see that $X\cap S_h=\varnothing$. Therefore, $\{F_1, \cdots, F_c\} \coprod \{X_i \mid h\nmid e_i\}$ form a regular sequence of R.

Case ii). In this case, the fixed points of the linear system $K[X_0, \cdots, X_n]_1$ is $V_+(X_0, \cdots, X_{n-1})$, which is the only singular point on $\boldsymbol{Q}(e)$. Therefore, by the definition of weighted complete intersection, the assertion is verified. Q.E.D.

The following example also satisfies the assumptions i) and ii) in Theorem 2.1.

Example 3.2. Let $n, q, e_0, \cdots, e_n, e_{n+1}, \cdots, e_{n+q}$ be positive integers with $n\geq 2$ and with $e_0=\cdots=e_n=1$. Let $K[X_0, \cdots, X_n, X_{n+1}, \cdots, X_{n+q}]$ be a graded ring whose gradation is given by $\deg X_i = e_i$ $(0 \leq i \leq n+q)$. Let c be a non-negative integer with $n-c \geq 2$. Take homogeneous elements $F_1, \cdots, F_c, F_{c+1}, \cdots, F_{c+q}$ in $K[X_0, \cdots, X_{n+q}]$ of degree>1, which satisfy the following conditions (1) and (2).

(1) $F_j \in K[X_0, \cdots, X_n]$ $(1 \leq j \leq c)$, and $F_1, \cdots, F_c$ form a regular sequence of $K[X_0, \cdots, X_n]$.

(2) $F_{c+j} = F'_{c+j} + X_{c+j}^{b_j}$ where $F'_{c+j} \in K[X_0, \cdots, X_{n+j-1}]$ and b_j is a positive integer $(1 \leq j \leq q)$.

Put $Y_j = \operatorname{Proj}(K[X_0, \cdots, X_{n+j}]/(F_1, \cdots, F_{c+j}))$ $(0 \leq j \leq q)$ and put $Y = Y_0$ and $X = Y_q$. Then it is easy to see that the following properties (3) and (4) hold.

(3) Y is a complete intersection of dimension≥ 2, and Y_j is a weighted complete intersection $(1 \leq j \leq q)$.

(4) The natural morphism $Y_j \to Y_{j-1}$, corresponding to the inclusion

$$K[X_0, \cdots, X_{n+j-1}] \hookrightarrow K[X_0, \cdots, X_{n+j}],$$

is a b_j-sheeted branched cyclic covering $(1 \leq j \leq q)$.

Changing variables $X_0, \cdots, X_n$ by a linear transformation if necessary, we see that the above example X satisfies the assumptions i) and ii) in Theorem 2.1.

Remark 3.3. Let Y be a complete intersection of dimension≥ 3, and let

$$X = Y_q \longrightarrow Y_{q-1} \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0 = Y$$

be a sequence of branched cyclic coverings over Y, that is, $Y_j \to Y_{j-1}$ is a branched cyclic covering $(1 \leq j \leq q)$.

Then, by virtue of (3) in § 1, we see that, as in Example 3.2, such X becomes a weighted complete intersection (cf. Corollary to Theorem 1.2 in [8]).

In case $\dim Y = 2$, the above statement is false in general (e.g. take $Y = \boldsymbol{P}^1 \times \boldsymbol{P}^1$).

Next we show some examples for which we can not apply our method.

Before showing examples, we prove a lemma.

Lemma 3.4. *Let Y be a non-singular projective algebaic variety over a field K and let Z be a non-singular closed subsheme of Y defined over K. Let $i: Y \hookrightarrow \boldsymbol{P}^N$ be a projective embedding. Assume that* $\dim Z < \frac{1}{2} \dim Y$.

Then there exists a hypersurface H in $\boldsymbol{P}^N$ with the following properties.

i) *$Y \cdot H$ is non-singular, and*

ii) *$Y \cdot H$ contains Z.*

Moreover, we may assume that, for such a hypersurface H, $\deg H$ *is large enough.*

Proof. Let $\mathscr{I}_Z$ be the $\mathcal{O}_Y$-ideal of Z. Let

$$\tilde{Y} = \operatorname{Proj}(\oplus_{n \geq 0} \mathscr{I}_Z^n) \xrightarrow{\pi} Y$$

be the blowing-up of Y with the center Z. Let $L=i^*\mathcal{O}_P(1)$ be the induced line bundle on Y. Then it is seen that, for a large integer a, $\pi^*(L^{\otimes a})\otimes\mathcal{O}_{\tilde{Y}}(1)$ is very ample. Let $j\colon \tilde{Y}\hookrightarrow P^{N'}$ be the projective embedding by the global sections of $\pi^*(L^{\otimes a})\otimes\mathcal{O}_{\tilde{Y}}(1)$. Note that each fibre of π is embedded linearly by j. We denote by B the set $\{H' \mid H'$ is a hyperplane of $P^{N'}$, which contains some fibres of $\pi^{-1}(Z)\to Z\}$. B can be considered as a closed subscheme of $\mathrm{Gr}(N', N'-1)$. Since $\pi^{-1}(Z)\to Z$ is a P^r-bundle, where $r=\operatorname{codim}_Y Z-1$, Z can be considered as a closed subscheme of $\mathrm{Gr}(N', r)$. Now consider the following diagram.

$$\begin{tikzcd}[column sep=small] & Fl(N'; N'-1, r) \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathrm{Gr}(N', N'-1)\supset B & & \mathrm{Gr}(N', r)\supset Z. \end{tikzcd}$$

By definition, we see that $B=p\circ q^{-1}(Z)$. Since q is a $P^{N'-(r+1)}$-bundle, we see that

$$\begin{aligned}\dim p\circ q^{-1}(Z) &\leq \dim q^{-1}(Z)=N'-(r+1)+\dim Z\\ &=N'-(\operatorname{codim}_Y Z-\dim Z)<N',\end{aligned}$$

where the last inequality follows from the assumption $\dim Z<\frac{1}{2}\dim Y$. Hence there is a hyperplane H' of $P^{N'}$ which satisfies the following conditions.

i′) $\tilde{Y}\cdot H'$ is non-singular, and

i″) H' does not contain any fibres of $\pi^{-1}(Z)\to Z$.

Since $\mathcal{O}_Y(\pi(H'))=L^{\otimes a}$, there is a hypersurface H of P^N of degree a with $Y\cdot H=\pi(H')$. It is easy to see that this hypersurface H possesses the required properties. Q.E.D.

The following example shows that if we admit only the assumption i) in Theorem 2.1, our argument in proving the local Torelli theorem breaks down.

Example 3.5. Let e_0, e_1, e_2, e_3, e_4 be positive integers. Assume that

(1) $e_0=e_1=1$, and that

(2) e_2, e_3, e_4 are greater than 1 and any two of them is mutually prime. Let $R=C[X_0, \cdots, X_4]$ be the graded ring with $\deg X_i=e_i$ $(0\leq i\leq 4)$. Put $m=\text{l.c.m.}\{e_0, \cdots, e_4\}=e_2e_3e_4$, and let

$$F_1=X_0^m+X_1^m+e_2X_2^{m/e_2}+e_3X_3^{m/e_3}+e_4X_4^{m/e_4}$$

be a homogeneous element in R of degree m. Put $Y=\operatorname{Proj}(R/(F_1))$ and $Z=\operatorname{Proj}(R/(F_1, X_0, X_1))$. Then it is easy to see by the Jacobian criterion that Y and Z are non-singular weighted complete intersections in $P(e_0, \cdots, e_4)$ of type (m) and of type $(m, 1, 1)$ respectively.

Applying Lemma 3.4 to Y, Z and $L=\mathcal{O}_Y(m)$, we have a homogeneous element F_2 in R of degree am for a large enough integer a, satisfying the following conditions.

(3) $X=\operatorname{Proj}(R/(F_1, F_2))$ is a non-singular weighted complete intersection in $\boldsymbol{P}(e_0, \cdots, e_4)$ of type (m, am).

(4) X contains Z.

For this X, it is easy to see the follows.

(5) $H^0(X, \Omega_X^1(\ell)) \neq 0$ ($\ell = e_2, e_3, e_4, e_3e_4, e_4e_2, e_2e_3$) (cf. the proof of Lemma 2.2).

(6) F_1, F_2, X_0, X_1 do not form a regular sequence of R.

The fact (5) shows that if we want to prove the local Torelli theorem for this X by our method, we shall have to begin cancelling by using X_0 and X_1. But it is impossible because of the fact (6) (cf. the proof of Theorem 2.1).

The next example shows that we can not apply our method, if we remove the assumption i) in Theorem 2.1.

Example 3.6. Let $e_0=5$, $e_1=7$, $e_2=8$ and $e_3=9$. Let X be a non-singular weighted hypersurface in $\boldsymbol{P}(e_0, e_1, e_2, e_3)$ (such a X exists). For this X, it is easy to see the following.

(1) $H^0(X, \Omega_X^1(\ell)) \neq 0$ ($\ell = e_ie_j$, $i \neq j$) (cf. the proof of Lemma 2.2). Hence, by using any pair of $\{X_i \mid 0 \leq i \leq 3\}$, we can not begin canceling (cf. the proof of Theorem 2.1).

References

[1] P. Deligne, Cohomologie des intersections complètes, exp. XI in S.G.A. 7 II, Springer Lecture Notes, **340** (1973).

[2] P. A. Griffiths, Periods of integrals on algebraic manifolds I, II, III, Amer. J. Math., **90** (1968), 568–626; 805–865; Publ. Math. I.H.E.S., **38** (1970).

[3] ——, Periods of integrals on algebraic manifolds: Summary of main results and discussion of open problems, Bull. Amer. Math. Soc., **75** (1970).

[4] P. A. Griffiths and W. Schmid, Recent developments in Hodge theory: a discussion of techniques and results, Proc. Symp. at Bombey in 1973, Oxford Univ. Press, 1975.

[5] S. Mori, On a generalization of complete intersections, J. Math. Kyoto Univ., 15-3 (1975).

[6] C. Peters, The local Torelli theorem of some cyclic branched coverings, to appear.

[7] S. Usui, The local Torelli theorem for non-singular complete intersections, Japanese J. Math., 2-2 (1976).

[8] J. J. Wavorik, Deformations of branched coverings of complex manifolds, Amer. J. Math., **90** (1968).

KYOTO UNIVERSITY